IIT-JEE 2009 MATHEMATICS - Paper - I

SECTION - I
Single Correct Choice Type

21. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is

(A) 48 (B) 32 (C) 40 (D) 80

21. **(A)** $z\bar{z}(z^2 + \bar{z}^2) = 350 \Rightarrow x^4 - y^4 = 175 \quad \therefore \quad (x, y) \equiv (\pm 4, \pm 3) \quad \therefore$ Area = 48.

22. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1$ and $\vec{a} \cdot \vec{c} = \frac{1}{2}$, then

(A) $\vec{a}, \vec{b}, \vec{c}$ are non coplanar
(B) $\vec{b}, \vec{c}, \vec{d}$ are non coplanar
(C) $\vec{b}, \vec{d}$ are non parallel
(D) $\vec{a}, \vec{d}$ parallel and $\vec{b}, \vec{c}$ are parallel

22. **(C)** Since unit vectors, only possible if $|(\vec{a} \times \vec{b})| = |(\vec{c} \times \vec{d})| = 1$

$\Rightarrow$ $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{c}$ and $\vec{d}$ are perpendicular and $(\vec{a} \times \vec{b}) = (\vec{c} \times \vec{d}) = \hat{n}$

$\therefore$ all lie in a plane normal to $\hat{n}$. Since $\vec{a}$ and $\vec{c}$ are non perpendicular

Hence, no two vectors are parallel.

23. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin.

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

23. **(D)** $M\left(-\frac{12}{5}, \frac{9}{5}\right) \Rightarrow$ Area = $\frac{27}{10}$ sq. units.

24. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} Im(z^{2m-1})$ at $\theta = 2°$ is

(A) $\frac{1}{\sin 2°}$ (B) $\frac{1}{3\sin 2°}$ (C) $\frac{1}{2\sin 2°}$ (D) $\frac{1}{4\sin 2°}$

24. **(D)** $\sin\theta + \sin 3\theta + \ldots\ldots + \sin 29\theta = \frac{1}{4\sin 2°}$.

25. Let $P(3, 2, 6)$ be a point in space and Q be a point on the line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

25. **(A)** $\overrightarrow{PQ} = (-3\mu - 2)\hat{i} + (\mu - 3)\hat{j} + (5\mu - 4)\hat{k}$

$\therefore \quad \overrightarrow{PQ}.(\hat{i} - 4\hat{j} + 3\hat{k}) = 0 \quad \Rightarrow \quad \mu = \frac{1}{4}$.

26. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(A) 55 (B) 66 (C) 77 (D) 88

26. **(C)** Coeff. of k^{10} in $(k + k^2 + k^3)^7$ = Coeff. of k^3 in $(1 + k + k^2)^7 = {}^7C_1 \times {}^6C_1 + {}^7C_3 = 77$.

27. Let f be a non negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt$, $0 \le x \le 1$, and $f(0) = 0$, then

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

27. **(C)** After differentiating the given equation, we get $f(x) = \sin x$, $\sin x < x \ \forall \ x > 0$.

28. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$ (B) $x^2 + y^2 - 4x - 10y + 19 = 0$

(C) $x^2 + y^2 - 2x + 6y - 29 = 0$ (D) $x^2 + y^2 - 6x - 4y + 19 = 0$

28. **(B)** points P, A, B, C (where C is the centre) are concyclic with PC as diameter.

$\Rightarrow$ $(x - 1)(x - 3) + (y - 2)(y - 8) = 0$.

SECTION - II
Multiple Correct Choice Type

29. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2 \frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then

(A) $b + c = 4a$ (B) $b + c = 2a$

(C) locus of point A is an ellipse (D) locus of point A is a pair of straight lines

29. **(BC)** $\cos((B - C)/2) = 2\sin(A/2)$ $\Rightarrow$ $\sin B + \sin C = 2\sin A$

$\Rightarrow$ $b + c = 2a$.

$\Rightarrow$ Since, BC is fixed. Hence, locus of A is an ellipse with B and C as two focii.

30. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then

(A) $\tan^2 x = \frac{2}{3}$ (B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$ (D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

30. **(AB)** $\tan^2 x = t$ $\Rightarrow$ the given equation is $(3t^2 - 2)^2 = 0$

$\therefore$ $\tan^2 x = \frac{2}{3}$. Using this we get, $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$.

31. Let $L = \lim_{x\to 0} \dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then

(A) $a = 2$ (B) $a = 1$ (C) $L = \dfrac{1}{64}$ (D) $L = \dfrac{1}{32}$

31. **(AC)** Using L.H.Rule, $a = 2$ and required limit = $L = \lim_{t\to 2} \dfrac{(2-t)}{8t(4-t^2)}$,

where $t = \sqrt{(a^2 - x^2)}$ $\therefore$ $L = \dfrac{1}{64}$.

32. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(A) $e - 1$ (B) $\int_1^e ln(e+1-y)dy$

(C) $e - \int_0^1 e^x\,dx$ (D) $\int_1^e ln\,y\,dy$

32. **(BCD)** Actual area = $\int_0^1 (e - e^x)\,dx = \int_1^e ln\,y\,dy = 1$.

SECTION - III
Linked Comprehension Type

Paragraph for Question Nos. 33 to 35.

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

33. The probability that $X = 3$ equals

(A) $\dfrac{25}{216}$ (B) $\dfrac{25}{36}$ (C) $\dfrac{5}{36}$ (D) $\dfrac{125}{216}$

33. **(A)** $X = 3$, required probability = $\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{1}{6} = \dfrac{25}{216}$.

34. The probability that $X \geq 3$ equals

(A) $\dfrac{125}{216}$ (B) $\dfrac{25}{36}$ (C) $\dfrac{5}{36}$ (D) $\dfrac{25}{216}$

34. **(B)** $X \geq 3$, required probability $= \frac{25}{216}\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + ...\infty\right) = \frac{25}{36}$.

35. The conditional probability that $X \geq 6$ given $X > 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{216}$ (C) $\frac{5}{36}$ (D) $\frac{25}{36}$

35. **(D)** $X > 3$, probability $= \frac{125}{1296} \times 6 = \frac{125}{216}$; $X \geq 6$, probability $= \frac{5^5}{6^5} \times \frac{1}{6} \times 6 = \frac{5^5}{6^5}$.

$\therefore$ Required probability $= \frac{5^5 / 6^5}{5^3 / 6^3} = \frac{25}{36}$.

Paragraph for Question Nos. 36 to 38.

Let $\mathscr{A}$ be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

36. The number of matrices in $\mathscr{A}$ is

(A) 12 (B) 6 (C) 9 (D) 3

36. **(A)** Either all diagonal entries are 1 or diagonal has only one 1.
$\therefore$ $n = 1$ (diagonals) * ${}^3C_1$ (for selecting place for 1)
+ ${}^3C_1$ (place for 1 in diagonal) * ${}^3C_1$ (for selecting place for 0) = 12.

37. The number of matrices A in $\mathscr{A}$ for which the system of linear equation

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution, is

(A) less than 4 (B) at least 4 but less than 7

(C) at least 7 but less than 10 (D) at least 10

37. **(B)** Unique solution if matrix is not singular. If row-1 be p, a, b and diagonal elements be p, q, r, then $D = det\ A = pqr + 2abc - a^2r - b^2q - c^2p$.
Here $abc = 0$ necessarily. If $pqr = 1$, exactly one of a, b or c is 1, giving $D = 0$. If $p = 1$ & $q = r = 0$, then non-singular only if $c = 1$ and in a & b one is 0 : 2 cases.
Similarly for $q = 1$ and $r = 1$ giving a total of 6 solutions.

38. The number of matrices A in $\mathscr{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent, is

(A) 0 (B) more than 2 (C) 2 (D) 1

38. **(B)** Inconsistent if $D = 0$ and $D_x \neq 0$. $\therefore\ qr \neq c^2$.
If $c = 0$, $q = r = 1$, then $p = 1$ also. For $a = 1$ or $b = 1$, $D = 0$: 2 solutions.
If $c = 1$ & $q = r = 0$, then $p = 1$ giving $D \neq 0$: not possible.
If $c = 1$ & $q + r = 1$, then $p = 0$ and $a = q$, $b = r$: 2 solutions.
$\therefore$ Total solutions = 4.

SECTION - IV
Matrix Match Type

39. Match the conics in **Column I** with the statements/expression in **Column II**

Column I	Column II
(A) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touch the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\lvert z + 2 \rvert - \lvert z - 2 \rvert = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \leq x < \infty$
	(t) Points z in the complex plane satisfying $Re\,(z + 1)^2 = \lvert z \rvert^2 + 1$

39. **(A) → (p); (B) → (st); (C) → (r); (D) → (qs)**
For (p): Required locus is $x^2 + y^2 = 1 / 4$. $\Rightarrow$ Circle.
For (q): Required locus is $\lvert PS - PS' \rvert$ = constant $\Rightarrow$ Hyperbola.
For (r): $x = \sqrt{3}\cos 2\theta$, $y = \sin 2\theta$, where $t = \tan\theta$. $\therefore$ $(x^2 / 3) + y^2 = 1$ $\Rightarrow$ Ellipse.
For (s): $e = 1$, parabola, $e > 1$: Hyperbola.
For (t): $Re\,(z + 1)^2 = \lvert z \rvert^2 + 1$ $\Rightarrow$ $y^2 = x$ $\Rightarrow$ Parabola.

40. Match the statements/ expressions in **Column I** with the open intervals in **Column II**

Column I		**Column II**	
(A)	Interval contained in the domain of definition on non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$	(p)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B)	Interval containing the value of the intergal $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q)	$\left(0, \frac{\pi}{2}\right)$
(C)	Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r)	$\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
		(s)	$\left(0, \frac{\pi}{8}\right)$
(D)	Interval in which $\tan^{-1}(\sin x + \text{con } x)$ is increasing	(t)	$(-\pi, \pi)$

40. **(A) → (pqs); (B) → (pt); (C) → (pqrt); (D) → (s)**

For (A): Since $ln\ y = 1/(x-3) + c$. Hence, domain: $R - \{3\}$.

For (B): $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx = \int_0^4 x(x-1)(x-2)(x-3)(x-4)\,dx$

$= \int_{-2}^{2} (x+2)(x+1)x(x-1)(x-2)dx = 0,$

Since $f(x) = (x+2)(x+1)x(x-1)(x-2)$ is odd.

For (C): $\sin 2x = \cos x \Rightarrow x = \pi/6$.

For (D): $f'(x) = (\cos x - \sin x)/(2 + \sin 2x) > 0$ if $\cos x > \sin x \therefore x \in (0, \pi/4)$.