

PART II - Mathematics

SECTION - I

Single Correct Choice Type

20. A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point Q . The length of the line segment PQ equals
(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2
20. (C) $(2 + a, -1 + a, 2 + a)$ on $2x + y + z = 9$ for $a = 1 \quad \therefore l = \sqrt{3} \quad a = \sqrt{3}$.
21. The normal at a point P on the ellipse $x^2 + y^2 = 16$ meets the x -axis at Q . If M is the mid point of the line segment PQ , then the locus of M intersects the latus rectums of the given ellipse at the points
(A) $\left(\pm \frac{3\sqrt{5}}{2}, \pm \frac{2}{7}\right)$ (B) $\left(\pm \frac{3\sqrt{5}}{2}, \pm \frac{\sqrt{19}}{4}\right)$
(C) $\left(\pm 2\sqrt{3}, \pm \frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm \frac{4\sqrt{3}}{7}\right)$
21. (C) $P(4 \cos \theta, 2 \sin \theta)$, $m_{PQ} = 2 \tan \theta$, $Q(3 \cos \theta, 0)$
 $M(\cos \theta / 2, \sin \theta) \Rightarrow (2x / 7)^2 + y^2 = 1$ where $x = \pm 2\sqrt{3}$, $y = \pm 1 / 7$.
22. If the sum of first n terms of an A.P. is cn^2 , then the sum of squares of these n terms is
(A) $\frac{n(4n^2 - 1)c^2}{6}$ (B) $\frac{n(4n^2 + 1)c^2}{3}$ (C) $\frac{n(4n^2 - 1)c^2}{3}$ (D) $\frac{n(4n^2 + 1)c^2}{6}$
22. (C) $T_n = c(n^2 - (n - 1)^2) = c(2n - 1)$
 $\therefore c^2 \left(\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \right) = \frac{nc^2}{3}(4n^2 - 1)$.

23. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$, and $y = 0$, where $p \neq q$, is
 (A) a hyperbola (B) a parabola (C) an ellipse (D) a straight line
23. (D) Vertices $(p, 0)$, $(q, 0)$.
 \therefore Altitudes: $px + (1 + p)y = pq$ and $qx + (1 + q)y = pq$
 \Rightarrow Orthocentre: $(-pq, pq) \therefore$ Locus is $y + x = 0$.

SECTION - II

Multiple Correct Choice Type

24. For the function $f(x) = x \cos \frac{1}{x}$, $x \geq 1$,
 (A) for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
 (B) $\lim_{x \rightarrow \infty} f'(x) = 1$
 (C) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
 (D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$
24. (AB) $f'(x) = \cos \frac{1}{x} - \frac{1}{x} \sin \frac{1}{x}$
 As $x \uparrow$, $1/x \downarrow$, $\sin(1/x) \downarrow$, $\cos(1/x) \uparrow$, $f'(x) \uparrow$.
 For $x \rightarrow \infty$, $f'(x) \rightarrow \cos 0 - 0 \sin 0 = 1 \Rightarrow f'(x) < 1 \forall x$
 By LMVT, $\frac{f(x+2) - f(x)}{(x+2) - x} = f'(c) < 1 \Rightarrow f(x+2) - f(x) < 2 \forall x$.
25. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at the point P on it meet its axis at points T and N , respectively. The locus of the centroid of the triangle PTN is a parabola whose
 (A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is $x = 0$
 (C) latus rectum is $\frac{2a}{3}$ (D) focus is $(a, 0)$
25. (AD) $P(at^2, 2at)$, $T(-at^2, 0)$, $N(2a + at^2, 0)$
 $\therefore 3x = 2a + at^2$ and $3y = 2at \therefore 3x - 2a = a(3y/2a)^2$ or, $9x = 12ax - 8a^2$
 $\Rightarrow y^2 = 4(a/3)(x - (2a/3))$
 \therefore Vertex: $(2a/3, 0)$, directrix: $x = a/3$, latus rectum = $4a/3$ and focus is $(a, 0)$.

26. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx$, $n = 0, 1, 2, \dots$, then

- (A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$
 (C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

26. (ABC) Replacing x with $-x$ and adding, $I_n = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$

$$\therefore I_n - I_{n-2} = \int_0^{\pi} \frac{\sin nx - \sin(n-2)x}{\sin x} dx = \int_0^{\pi} 2 \cos(n-1)x dx = 0$$

Also, $I_1 = \pi$ and $I_0 = 0 \Rightarrow I_{\text{odd}} = \pi$ and $I_{\text{even}} = 0$.

27. For $0 < \theta < \pi/2$, the solution(s) of $\sum_{m=1}^6 \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is
 (are)
 (A) $\pi/4$ (B) $\pi/6$ (C) $\pi/12$ (D) $5\pi/12$

27. (CD) Using $\operatorname{cosec} \alpha \operatorname{cosec} \beta = (\cot \beta - \cot \alpha) / \sin(\alpha - \beta)$

$$\therefore \sum_{m=1}^6 \cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2} \sin \frac{\pi}{4} = 4$$

$$\Rightarrow \cot \theta - \cot\left(\theta + \frac{6\pi}{4}\right) = 4 \Rightarrow \cot \theta + \tan \theta = 4 \Rightarrow \cot \theta = 2 \pm \sqrt{3}.$$

28. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then
 (A) equation of ellipse is $x^2 + 2y^2 = 2$ (B) the foci of ellipse are $(\pm 1, 0)$
 (C) equation of ellipse is $x^2 + 2y^2 = 4$ (D) the foci of ellipse are $(\pm \sqrt{2}, 0)$

28. (AB) For orthogonal intersection, foci coincide.
 \therefore For ellipse, $e = 1/\sqrt{2}$ and foci $(\pm 1, 0)$
 $\Rightarrow a = \sqrt{2}, b = 1 \Rightarrow$ Ellipse is $x^2 + 2y^2 = 2$.

SECTION - III
Matrix Match Type

29. Match the statements/expressions given in **Column I** with the values in **Column II**

Column I

Column II

- | | |
|---|---------------|
| (A) Root(s) of the equation
$2 \sin^2 \theta + \sin^2 2\theta = 2$ | (p) $\pi / 6$ |
| (B) Points of discontinuity of the equation
$f(x) = \left[\frac{6x}{\pi} \right] \cos \left[\frac{3x}{\pi} \right]$, where $[y]$ denotes
the largest integer less than or equal to y | (q) $\pi / 4$ |
| (C) Volume of the parallelopiped with its edges
represented by the vectors $\hat{i} + \hat{j}$, $\hat{i} + 2\hat{j}$
and $\hat{i} + \hat{j} + \pi\hat{k}$ | (r) $\pi / 3$ |
| (D) Angle between vectors \vec{a} and \vec{b} where \vec{a} , \vec{b}
and \vec{c} are unit vectors satisfying
$\vec{a} + \vec{b} + \sqrt{3}\vec{c} = \vec{0}$ | (s) $\pi / 2$ |

(t) π

29. (A) \rightarrow (qs); (B) \rightarrow (prst); (C) \rightarrow (t); (D) \rightarrow (r)

- (A) $\sin^2 2\theta = 2 \cos^2 \theta \Rightarrow 2 \cos^2 \theta (1 - 2 \sin^2 \theta) = 0$
 $\Rightarrow \cos \theta = 0$ or $\cos 2\theta = 0 \Rightarrow \pi / 4, \pi / 2$.
- (B) At $x = k\pi / 6$, $[6x / \pi]$ is discontinuous and $\cos [3x / \pi]$ is non zero.
 $\therefore \pi / 6, \pi / 3, \pi / 2, \pi$.
- (C) $(\hat{i} + \hat{j}) \times (\hat{i} + 2\hat{j}) = k \quad \therefore \text{STP} = k \cdot (\hat{i} + \hat{j} + \pi\hat{k}) = \pi$.
- (D) $\vec{a} + \vec{b}$ has magnitude $\sqrt{3}$ only if angle between \vec{a} and \vec{b} is $\pi / 3$.

30. Match the statements/ expressions in **Column I** with the open intervals in **Column II**

Column I

Column II

(A) The number of solutions of the equation (p) 1

$$xe^{\sin x} - \cos x = 0 \text{ in the interval } \left(0, \frac{\pi}{2}\right)$$

(B) Value(s) of k for which the planes (q) 2

$$kx + 4y + z = 0, 4x + ky + 2z = 0 \text{ and } 2x + 2y + z = 0 \text{ intersect in a straight line}$$

(C) Value(s) of k for which (r) 3

$$|x - 1| + |x - 2| + |x + 1| + |x + 2| = 4k$$

has integer solution(s)

(s) 4

(D) If $y' = y + 1$ and $y(0) = 1$, then the value(s) (t) 5

of $y(\ln 2)$

30. (A) \rightarrow (p); (B) \rightarrow (ps); (C) \rightarrow (qrst); (D) \rightarrow (r)

(A) $xe^{\sin x} - \cos x$ is increasing function in $\left(0, \frac{\pi}{2}\right)$.

Since increases from -1 to $\pi e / 2$, exactly 1 solution.

$$(B) \begin{vmatrix} k & 4 & 1 \\ 4 & k & 2 \\ 2 & 2 & 1 \end{vmatrix} = 0 \Rightarrow k^2 + 8 - 6k = 0 \Rightarrow k = 2 \text{ or } 4.$$

(C) Expr. has minimum value = 6 for $-1 \leq x \leq 1$

$$\Rightarrow 4k \geq 6.$$

For $k \geq 2$, solution of $x = k$ is valid.

$$(D) \ln(y + 1) = x + c = x + \ln 2 \Rightarrow y = 2e^x - 1 \Rightarrow y(\ln 2) = 3.$$

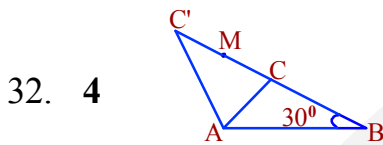
SECTION - IV

Integer Answer Type

31. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C , then the radius of the circle C is

31. **8** If centres be $(-3, 0)$ & $(3, 0)$, P be $(0, 0)$, common tangent be $x - \sqrt{8}y = 0$, for C ; if centre be $(0, a)$ and radius r , then $\sqrt{8}a / 3 = r$ and $9 + a^2 = (r + 1)^2 \Rightarrow 8$.

32. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30^\circ$. The absolute value of the difference between the areas of these triangles is



$$AM = 4 \sin 30^\circ = 2 \quad \therefore \quad \angle CAM = \angle C'AM = 45^\circ$$

$\Delta C'AC$ is right angled with area = 4.

33. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$.

Then the value of $f(\ln 5)$ is

33. **0** $f'(x) = f(x)$ and $f(0) = 0 \Rightarrow f(x) = ae^x$

Since $f(0) = 0$, $f(x) = 0 \forall x \quad \therefore f(\ln 5) = 0$.

34. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations:

$$3x - y - z = 0$$

$$-3x + z = 0$$

$$-3x + 2y + z = 0.$$

Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is

34. **7** Solving $y = 0$ and $z = 3x \quad \therefore 10x^2 \leq 100 \Rightarrow |x| \leq 3$.

$\therefore x = \pm 3, \pm 2, \pm 1, 0$ i.e. 7 points.

35. If the function $f(x) = x^3 + e^{(x/2)}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is
35. **2** $g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$
 Putting $x = 0$, $g'(1) \cdot (1/2) = 1 \Rightarrow g'(1) = 2.$
36. The smallest value of k , for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is
36. **2** $D > 0$, $f(4) \geq 0$, $-b/2a > 4.$
 $\Rightarrow k - 1 > 0$, $(k - 1)(k - 2) \geq 0$, $4k > 4 \Rightarrow k_{\min} = 2.$
37. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \leq 9x\}$ is
37. **7** $x^2 - 9x + 20 \leq 0 \Rightarrow 4 \leq x \leq 5$
 $f'(x) = 6(x - 2)(x - 3) > 0$ for $4 \leq x \leq 5.$ \therefore Max. Value = $f(5) = 7.$
38. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \rightarrow 0} \left(1 + \frac{p(x)}{x^2}\right) = 2.$
- Then the value of $p(2)$ is
38. **0** $P(x) = x^2 + ax^3 + bx^4 \Rightarrow P'(x) = 2x + 3ax^2 + 4bx^3$
 Since extremum at $x = 1$ and $x = 2$, $2 + 3a + 4b = 0$ and $2 + 6a + 16b = 0$
 $\Rightarrow b = 1/4$, $a = -1 \Rightarrow P(x) = x^2 - x^3 + x^4/4$
 $\Rightarrow P(2) = 4 - 8 + 4 = 0.$